A Note on Curve Counting Scheme in an Algebraic Family and The Admissible Decomposition Classes

Ai-Ko Liu*

February 1, 2008

In this paper, we discuss the generalized scheme for curve counting in the family Seiberg-Witten theory. Even though the original motivation is to study the Mcduff's proposal in $b_2^+ = 1$ category of symplectic four manifolds, we will formulate our scheme in an algebraic(Kahler) set up. The material considered in this paper will be relatively elementary. Nevertheless, the theory discussed here has played an essential role in the long paper [Liu1].

As a major application of the discussion, one may apply our scheme in the proof of the Göttsche's conjecture about counting of holomorphic curves[Liu1]. The current scheme is motivated from the discussion [Mc] of pseudo-holomorphic curves in symplectic four manifolds. However we will restrict our discussion to the algebraic varieties here. One can translate our scheme to pseudo-holomorphic category by replacing holomorphic curves to pseudo-holomorphic curves.

Given an algebraic family of algebraic surfaces $\mathcal{X} \mapsto B$, the family Seiberg-Witten invariant $\mathcal{AFSW}$ (or FSW in the smooth category) enumerates the algebraic curves (or pseudo-holomorphic curves in the symplectic category) within the family dual to a given cohomology class. The basic phenomenon we will study is that not only smooth curves may appear in the enumeration, curves contain multiple coverings of the so-called exceptional curves may also occur. The general question we are interested at is
Question: How to relate the contribution from the smooth curves to the original family invariant?

The general strategy to answer the question is to subtract the contributions from the various configurations containing multiple coverings of exceptional curves. The purpose of the current paper is to provide a skeleton of the curve counting scheme.

*email address: akliu@math.berkeley.edu, Current Address: Mathematics Department of U.C. Berkeley

1 The family scheme of Algebraic Surfaces

In this section, consider $\mathcal{X} \mapsto B$ to be a relatively smooth algebraic fibration over a smooth base. For simplicity, the field of definition can be taken to be $\mathbf{C}$. The same scheme will work for any algebraic closed field of characteristic zero as well. Even though we do not aim at general symplectic four manifolds, we will recall the phenomena in the symplectic set up from time to time.

The fibers of the fibration are taken to be smooth projective surfaces. In the following, we denote $dim_{\mathbf{C}}B$ to be the complex dimension of the base even though the scheme work well even when B does not carry complex structures.

First we introduce certain notations. Let C denote the cohomology class represented by holomorphic curves. Let $d_{GT}(C) = \frac{C^2 - c_1(K_X)\cdot C}{2}$ denote the complex Gromov Taubes dimension [T3] in Taubes theory. Then the family Gromov-Taubes dimension of C is defined axiomatically to be $d_B(C) = d_{GT}(C) + dim_{\mathbf{C}}B$.

Given the class C as the sum of C_1 and C_2, one has the following equality between their formal family Gromov-Taubes dimensions.

$$d_B(C) = d_B(C_1) + d_B(C_2) - dim_{\mathbf{C}}B.$$

This equality reflects that the family moduli space associated to C is the fiber product of those of C_1 and C_2.

One can easily generalize the equality to more than two C_i and the new equality is

$$d_B(\sum_{i\in I} C_i) = \sum_{i\in I} d_B(C_i) - (|I| - 1)dim_{\mathbf{C}}B,$$

with $|I|$ being the cardinality of the index set I.

Definition 1 *A cohomology class $e \in H^2(M, \mathbf{Z})$ is said to be an exceptional class if*

(i). e is a primitive element in the lattice $H^2(M, \mathbf{Z})$.

(ii). $e^2 = e \cup e[M] < 0$.

A pseudo-holomorphic curve in an almost complex four-manifold M is said to be an exceptional curve if it is poincare dual to an exceptional class e.

We have the following proposition regarding irreducible exceptional curves,

Proposition 1 *Let Σ_1, Σ_2 be two Riemann surfaces and let $f_1 : \Sigma_1 \mapsto M$ $f_2 : \Sigma_2 \mapsto M$ be two (pseudo-)holomorphic maps into the almost complex four-manifold M with $(f_i)_*[\Sigma_i]$ dual to an exceptional class $e \in H^2(M, \mathbf{Z})$, then $\Sigma_1 = \Sigma_2$ and f_1 and f_2 coincide. Namely, $f_i(\Sigma_i), i = 1, 2$ coincide and is the only irreducible (pseudo-)holomorphic curve dual to e in M.*

Proof of the proposition: Because e is primitive, the maps f_i from Σ_i to $f_i(\Sigma_i)$ are of degree 1. From [Mc], $f_i(\Sigma_i), i = 1, 2$ have at most a finite number of

isolated singularities and f_i are immersions away from the singularities in the images. Because Σ_i are irreducible, $f_i(\Sigma_i)$ are irreducible, too.

We argue that $f_1(\Sigma_1) = f_2(\Sigma_2)$. If not, the sets $f_1(\Sigma_1) = \cap f_2(\Sigma_2)$ is of a finite cardinality. However, again by [Mc2] each intersection point contributes positively to the total intersection number, the total intersection number should be non-negative. On the other hand, $f_1(\Sigma_1) \cap f_2(\Sigma_2) = (f_1)_*[\Sigma_1] \cap (f_2)_*[\Sigma_2] = e \cup e[M] < 0$.

This gives us the necessary contradiction. Thus $f_1(\Sigma_1) = f_2(\Sigma_2)$. Once we know $f_1(\Sigma_1) = f_2(\Sigma_2)$ and both pseudo-holomorphic maps are of degree one, Both Σ_1, Σ_2 can be re-constructed as the normalization of the complex curve $f_1(\Sigma_1) = f_2(\Sigma_2)$. Therefore $\Sigma_1 = \Sigma_2$ and $f_1 = f_2$. $\square$

Even though the "irreducible" curve dual to e is always unique, there can be two or more reducible pseudo-holomorphic curves dual to e with more than one irreducible component. In this case, the conclusion is weaker and one can only deduce that two curves share at least one irreducible component and the fundamental class of this irreducible component has a negative self-intersection number in the four-manifold M.

Recall that Taubes theory [T3] asserts the equivalence of Seiberg-Witten invariant and a version of Gromov invariant for symplectic four manifolds. It indicates that the diffeomorphism invariants SW is equivalent to the symplectic invariant Gr. Despite of the simplicity of the statement, the actual proof [T1], [T2], [T3] involves sophisticated analysis and an amount of new ideas.

It is less well known that the equivalence of SW and Gr fails for the general symplectic four manifolds with $b_2^+ = 1$. Originally Taubes asserted his theorem for $b_2^+ > 1$ category. Later he extended his theorem to $b_2^+ = 1$ case with some additional assumption. In the mean time, it was discovered experimentally by the current author in [LL] that the assertion would not be true without the additional assumption. This motivated Mcduff to change the original definition of the Gromov invariant in order to match up with SW.

Definition 2 *In defining the Seiberg-Witten invariants of $b_2^+ = 1$ symplectic four-manifold M, the space of generic Riemannian metrics g and self-dual two forms μ on M are divided into chambers. Given a symplectic two form ω on M and a Riemannian metric g such that ω is self-dual with respect to g, the $(g, r\omega = \mu)$, $r \mapsto \infty$ determines a unique chamber, called the Taubes' chamber in the following discussion.*

Let us recall Taubes theorem for $b_2^+ = 1$ symplectic manifolds.

Theorem 1 *(Taubes) Let C be a cohomology class in $H^2(M, \mathbf{Z})$, with M being a symplectic four manifold with $b_2^+ = 1$. Assume additionally that $C \cdot S \geq -1$ for all spherical class S with $S^2 = -1$. Then the statement $SW(2C - K_M) = Gr(C)$ holds for C, where $SW(2C - K_M)$ is evaluated in the Taubes' chamber.*

By a spherical class S with $S^2 = -1$ one means that the class is represented by a S^2 with self intersection number -1.

In the following, we give a simple example that the theorem does not hold without modification for the classes violating this condition.

Examples 1 *Consider M to be $\mathbf{CP}^2\sharp\overline{\mathbf{CP}^2}$, the symplectic four manifold constructed by $\mathbf{CP}^2$ by blowing up one point in $\mathbf{CP}^2$. Let $H \in H^2(M, \mathbf{Z})$ denote the (pull-back of) the hyperplane class and let $E \in H^2(M, \mathbf{Z})$ denote the exceptional class.*

Take $C = 3H + 2E$,and a simple calculation shows that $d_{GT}(C) = 9 - 1 = 8$. Through the calculation of wall crossing formula it is easy to see that $SW(9H + 3E) = \pm 1$. On the other hand, we argue that a reasonable definition of Gromov invariant would be $Gr(3H + 2E) = 0$. The class $3H + 2E$ can not be represented by irreducible curves. The representatives are the disjoint union of a cubic curve dual to $3H$ along with the double covering of the exceptional curve E. Even though curves dual to $3H$ gives rise to nonzero invariant, the multiple covering of E has a negative Gromov dimension $d_{GT}(2E) = -1$. Thus the total Gromov-Taubes invariant should be $Gr(3H) \times 0 = 0$.

This simple example illustrates the subtlety to Taubes' theory. Even though the $SW(9H + 3E) \neq 0$ in the Taubes' chamber and the analysis in $SW \mapsto Gr$ [T1] still implies the existence of pseudo-holomorphic curves in C, numerically the curves are counted zero in the Gromov-Taubes theory. As similar type of phenomena appears for any non-minimal symplectic four manifolds with $b_2^+ = 1$, it becomes the major topological obstruction to identify SW and Gr. The way Taubes dealed with this problem is to rule out the classes C which potentially can be represented as disjoint unions of pseudo-holomorphic curves and some multiple coverings of exceptional -1 curves. This explains the extra condition $C \cdot S \geq -1$ in Taubes' theorem.

Even though it is not completely obvious. this ill symptom is closely related to the fact that $b_2^+ = 1$ symplectic manifolds are not of simple type in Taubes' chamber. In the fundamental paper of Taubes, symplectic four manifolds with $b_2^+ > 1$ were proved to be of simple type. Thus, the ill symptom does not occur to them.

On the other hand, it is a consequence of the wall crossing formula [LL] that the non-simple type of $b_2^+ = 1$ symplectic manifolds is directly related to the non-vanishing of the wall crossing numbers.

The primitive goal to develop the family Seiberg-Witten theory is to discuss the family Seiberg-Witten theory in Taubes' chambers (defined by a large perturbation of fiberwise self-dual symplectic forms). As a similar application of the family wall crossing formula implies the non-simple type-ness, one would expect that the similar failure of $SW = Gr$ would occur. Unlike the $B = pt$ case that -1 curves are the only pseudo-holomorphic curves which persist, the topological types of the exceptional curves which persist in the family are less restricted.

Viewed from a different angle, the major distinction between the standard Gromov-Witten theory and Gromov-Taubes theory lies in the fact that the latter theory does not restrict the topological types of the pseudo-holomorphic maps.

The combination of these two issues make the identification between FSW and FGr extremely difficult. I.e./ within a given family of symplectic manifolds and a fiberwise monodromy invariant class C, there can be a whole 'zoo' of exceptional curves which may appear in some pseudo-holomorphic curve representations of the class C. To make sense of Gromov-Taubes invariants, one has to deal with these exotic objects in a more systematical way.

Examples 2 *For the readers with a background of Gromov-Witten theory, it should be cautious not to think of Gromov-Taubes invariants as identical to the standard Gromov-Witten invariants. Besides the question of allowing disconnected domain curves, the expected Gromov-Taubes dimensions of multiple covering of exceptional curves are different from the dimension formulae of Gromov-Witten invariants viewed as multiple covering pseudo-holomorphic maps into the symplectic manifold.*

For simplicity, let $e \in H^2(M, \mathbf{Z})$ *be an exceptional class* $e^2 = -k, c_1(K_M) \cdot e = -2 + k$ *representing an exceptional sphere. Let* $m \in \mathbf{N}$ *be a positive integer, the expected Gromov-Taubes dimension of the class* me *is given by* $(-m^2 - m)k + 2m$.

On the other hand, the expected dimension of the pseudo-holomorphic maps from S^2 *to* M*, dual to* me*, (modulo diffeomorphisms on* S^2*) is given by*

$$2\{c_1(M) \cdot C - (g-1) \cdot dim_{\mathbf{C}} M + 3g - 3\} = 2\{-c_1(K_M) \cdot C + 2 - 3\} = -2mk + 2m - 2.$$

The former is quadratic with respect to m*, while the usual Gromov-Witten expected dimension is linear in* m.

Thus, even though we will still call pseudo-holomorphic curves dual to me *a multiple covering of exceptional curves in* e*, we advise the readers not to confuse them with the multiple covering in the sense of maps. Instead, it is wiser to think of it as* m *copies of pseudo-holomorphic curves dual to* e *sitting on top of each other.*

2 The Pointwise Calculation of Family Dimension

Firstly, let us review the original dimension count argument of Taubes.

Let (M, ω) be a symplectic four-manifold with a compatible almost complex structure and $C \in H^2(M, \mathbf{Z})$ be a cohomology class with a positive energy $C \cdot \omega > 0$. In Taubes' theory, he allows the pseudo-holomorphic curves to have more than one irreducible component.

Suppose that there is a pseudo-holomorphic curve poincare dual to C. Then there is a finite collection of Riemann surfaces $\Sigma_i, 1 \leq i \leq k$ and the pseudo-holomorphic maps $f_i : \Sigma_i \mapsto M$ such that $\sum_{i \leq k} (f_i)_* [\Sigma_i] \in H_2(M, \mathbf{Z})$ is poincare dual to C.

In case the map f_i is of degree m_i, we may write $PD((f_i)_*[\Sigma_i])$ as $m_i e_i$, $m_i \in \mathbf{N}$. Then we have the following equality

$$C = \sum_{i \leq k} m_i e_i.$$

Conversely, if C is written as $\sum_{i \leq k} m_i e_i$ and each of e_i is represented by an irreducible pseudo-holomorphic curve, then one takes the union of them (counting multiplicity) and represent C as a pseudo-holomorphic curve in M.

Taubes would like to study all the possible decompositions of C into different e_i which will survive under generic compatible almost complex structures perturbation of M.

One makes three additional assumptions on e_i,

(i). $e_i \cdot \omega > \mathcal{E}(M, \omega) > 0$ for some manifold dependent lower bound of harmonic energy.

(ii). The expected Gromov-Taubes dimension $d_{GT}(e_i) = \frac{e_i^2 - e_i \cdot c_1(\mathbf{K}_M) \cdot e}{2} \geq 0$.

(iii). $e_i \cdot e_j \geq 0$ for all $i \neq j$.

(iv). $e_i^2 + c_1(\mathbf{K}_M) \cdot e_i = 2g_{arith}(e_i) - 2 \geq -2$.

Definition 3 *Let $e_i, i \leq k$ be a finite number of classes in $H^2(M, \mathbf{Z})$ which satisfy $(i).,(ii).,(iii).$ and (iv). The expression $C = \sum_{i \leq k} m_i e_i, m_i \in \mathbf{N}$ is called a (cohomological) decomposition of C.*

The reason that one imposes (i) is because a pseudo-holomorphic curve dual to e_i always has a positive energy.

If an irreducible irreducible curve dual to e_i has negative Gromov-Taubes dimension $d_{GT} = \frac{e_i^2 - c_1(\mathbf{K}_M) \cdot e_i}{2} < 0$, then by Fredholm theory this type of curves may disappear after a generic perturbation of compatible almost complex structures on M, which does not have chance to contribute to the Gromov invariant defined by Taubes [T3].

From [Mc2], one knows that two distinct irreducible pseudo-holomorphic curves in an almost complex four-manifold M intersect positively. Thus $e_i \cdot e_j$, the sum of all the local intersection contribution, should be non-negative as well.

Because e_i is represented by an irreducible pseudo-holomorphic curve on an almost complex four-manifold M, it satisfies the adjunction formula with g_{arith} being the arithmetic genus of the curve.

Let C be written as $\sum_{i \leq k} m_i e_i$ with multiplicity $m_i \geq 1$ satisfying $(i).$, $(ii).$, $(iii).$ and (iv).

Then

$$2d_{GT}(C) = C^2 - C \cdot c_1(\mathbf{K}_M) = \big(\sum_{i \leq k} (e_i^2 - e_i \cdot c_1(\mathbf{K}_M))$$

$$+2 \sum_{i \neq j} m_i m_j e_i \cdot e_j + \sum_{i \leq k} ((m_i^2 - 1) e_i^2 + (1 - m_i) e_i \cdot c_1(\mathbf{K}_M))\big)$$

$$= 2\sum_{i\le k} d_{GT}(e_i)+2\sum_{i\ne j} m_i m_j e_i\cdot e_j+\sum_{i\le k}\big((m_i^2-m_i)e_i^2+(m_i-1)(e_i^2-e_i\cdot c_1(\mathbf{K}_M))\big).$$

By the assumption (iv). $e_i^2 + c_1(\mathbf{K}_M)\cdot e_i \geq -2$ and by (ii). $2d_{GT}(e_i) = e_i^2 - c_1(\mathbf{K}_M)\cdot e_i \geq 0$. Then we know that $e_i^2 \geq -1$. Suppose $e_i^2 = -1$. From (ii). again we get $-1 = e_i^2 \geq c_1(\mathbf{K}_M)\cdot e_i$. Then $-2 \geq e_i^2 + c_1(\mathbf{K}_M)\cdot e_i$ and $c_1(\mathbf{K}_M)\cdot e_i = -1$ as well. If it is the case, $g_{arith}(e_i) = 0$ and one can argue that the pseudo-holomorphic curve representing e_i must be a so-called -1 curve.

Because Taubes' goal is to develop a version of Gromov invariant which can be identified with $SW(2C - c_1(\mathbf{K}_M))$ (the $spin^c$ class $2C - c_1(\mathbf{K}_M)$ is in an additive notation), he is able to use the Seiberg-Witten simple typeness condition on $b_2^+ > 1$ symplectic four-manifolds and the blowup formula of Seiberg-Witten invariants to deduce $m_i = 1$ for all such -1 classes $e_i^2 = -1$.

Thus, one find that the last term in the expansion of $2d_{GT}(C)$ is always non-negative.

In order to count pseudo-holomorphic curves dual to C, one imposes $d_{GT}(C)$ number of generic points and require the pseudo-holomorphic curves dual to C to pass through these generic points.

From above we find that

$$d_{GT}(C) - \sum_{i\le k} d_{GT}(e_i) \geq 0.$$

If the difference $d_{GT}(C) - \sum_{i\le k} d_{GT}(e_i)$ is strictly positive, then by dimension reason there can be no pseudo-holomorphic curves in $\sum_{i\le k} e_i$ which pass through all these $d_{GT}(C)$ points, after one adopts the Fredholm argument to perturb the almost complex structures of M. In order $\sum m_i e_i$ contributes to the Gromov invariant, the non-negative sum $2\sum_{i\neq j} m_i m_j e_i\cdot e_j + \sum_{i\le k}\big((m_i^2 - m_i)e_i^2 + (m_i-1)(e_i^2 - e_i\cdot c_1(\mathbf{K}_M))\big)$ has to vanish term by term.

That is to say,

(a). $m_i = 1, \forall i \leq k$.

(b). $e_i\cdot e_j = 0, \forall i \neq j$.

(c). Each curve dual to e_i must be smooth.

In other words, distinct e_i cannot intersect. Each irreducible pseudo-holomorphic curve appears with multiplicity one.

One may develop the following 'philosophical' idea which helps to explain what happens.

(a)'. If $m_i > 1$ for some $e_i^2 > 0$, ideally one may choose two distinct pseudo-holomorphic curves dual to e_i and they intersect positively. Then the smoothing of all these intersection singular points produces an irreducible curve dual to $2e_i$, whose dimension $d_{GT}(2e_i) > d_{GT}(e_i) + d_{GT}(e_i)$. Continue in this fashion, Seiberg-Witten theory is expected to count irreducible curves in $m_i e_i$ rather than m_i copies of curves dual to e_i, which formally can be viewed as a degeneration from an irreducible multiplicity one curve dual to $m_i e_i$.

(b)'. If $e_i \cdot e_j > 0$ for some $i \neq j$, one may think of the smoothing of the $e_i \cdot e_j$ intersection points (counted with multiplicity) and consider (formally) a curve dual to $e_i + e_j$. Then the union of the curves dual to e_i and e_j can be thought as a degeneration of some irreducible curve dual to $e_i + e_j$ and $d_{GT}(e_i + e_j) > d_{GT}(e_i) + d_{GT}(e_j)$.

(c)'. If a curve dual to e_i develops certain singularities, it can be thought of a degeneration of the smooth curves dual to e_i satisfying the adjunction equality $e_i^2 + c_1(\mathbf{K}_M) \cdot e_i = 2g(e_i) - 2$. The curves with singularities are of lower expected dimension than the expected Gromov-Taubes dimension $d_{GT}(e_i)$.

In the $b_2^+ = 1$ category, $m_i \geq 1$ for the -1 classes $e_i^2 = -1$. Then the same argument breaks down.

One way to remedy is to impose extra condition on C, as was done by Taubes (see the statement of theorem. 1).

Mcduff introduces a different way to remedy the situation. She (see [Mc]) has shown that

Proposition 2 *(Mcduff) Let M be a $b_2^+ = 1$ symplectic four-manifold and let $C \in H^2(M, \mathbf{Z})$ be a class satisfying $d_{GT}(C) = \frac{C^2 - C \cdot c_1(\mathbf{K}_M)}{2} \geq 0$, $C \cdot \omega > 0$. Then there exists a finite number of -1 classes, e_i, $e_i^2 = -1$ satisfying the following conditions:*

(i). Each e_i is represented by a -1 pseudo-holomorphic curve.

(ii). $e_i \cdot e_j = 0$ for $i \neq j$.

(iii). $C \cdot e_i = -n_i < 0$.

Then one may re-write $C = (C - \sum n_i e_i) + \sum n_i e_i$ and $C - \sum n_i e_i = C_{red}$ is perpendicular to all these e_i, namely $C_{red} \cdot e_i = 0$.

Mcduff proposed [Mc] to define $Gr(C)$ using the class C_{red} instead of C.

Definition 4 *Let $C = \sum_i m_i e_i$ be a cohomological decomposition of C. The decomposition is said to be of Taubes' type if*

(a). $e_i \cdot e_j = 0$ for $i \neq j$.

(b). $m_i = 1$ for all i.

In the following, we generalize the Mcduff's proposal to the family case. It turns out all the different possibilities of decompositions of curves appearing in the $B = pt$ case can also appear in the family case. Moreover, in the family case there are many new possible decompositions of curve classes which are absent in the $B = pt$ case due to dimension reason.

Let $\pi : \mathcal{X} \mapsto B$ be a fiber bundle of symplectic four-manifolds with the relative symplectic form $\omega_{\mathcal{X}/B}$.

Similar to the condition $(i).(ii).(iii).$ and $(iv).$ for the $B = pt$ cases, one imposes the following conditions on the classes e_i.

(Fi).$e_i \cdot \omega_{\mathcal{X}/B} > \mathcal{E}(\mathcal{X}, \omega_{\mathcal{X}/B}) > 0$ for some fiber bundle dependent lower bound of harmonic energy.

(Fii).The expected family Gromov-Taubes dimension $dim_{\mathbf{C}}B + d_{GT}(e_i) = \frac{e_i^2 - e_i \cdot c_1(\mathbf{K}_M) \cdot e}{2} \geq 0$.

(Fiii). $e_i \cdot e_j \geq 0$ for all $i \neq j$.

(Fiv). $e_i^2 + c_1(\mathbf{K}_{\mathcal{X}/B}) \cdot e_i = 2g_{arith}(e_i) - 2 \geq -2$.

At this moment we are doing the pointwise analysis for different $b \in B$, we do not take into account the monodromy action of $\pi_1(B, b_0) \mapsto H^2(\pi^{-1}(b_0), \mathbf{Z})$. At times (e.g. for the universal families $M_{l+1} \mapsto M_l$), the monodromy representation is completely trivial and we can ignore it. Otherwise, we have to consider the equivalent classes of e_i or decompositions under the action of $\pi_1(B, b_0) \mapsto H^2(\pi^{-1}(b_0), \mathbf{Z})$.

Let us discuss how does the ordinary Taubes' dimension count argument generalized to the family case. Because we will use the relative canonical bundle $\mathbf{K}_{\mathcal{X}/B}$ throughout the discussion, we will skip the subscript $\mathcal{X}/B$ and denote it by $\mathbf{K}$.

Using the family dimension formula,

$$dim_{\mathbf{R}}B + 2d(C) = dim_{\mathbf{R}}B + C^2 - C \cdot c_1(\mathbf{K}) = (\sum_{i \leq k}(dim_{\mathbf{R}}B + e_i^2 - e_i \cdot c_1(\mathbf{K})) - (k-1)dim_{\mathbf{R}}B$$

$$+2\sum_{i \neq j} m_i m_j e_i e_j + \sum_i ((m_i^2 - 1)e_i^2 + (1 - m_i)e_i \cdot K)).$$

In a given family, the condition (Fii). is weaker than the condition (ii). at the $B = pt$ case. Thus, there may be some e_i with $e_i^2 < -1$. Then the last expression $\sum_i((m_i^2 - 1)e_i^2 + (1 - m_i)e_i \cdot K))$ may be negative for $m_i \neq 1$. In other words, the formal dimension expected from family Seiberg-Witten theory (equal to $dim_{\mathbf{R}}B + 2d_{GT}(C)$ can be smaller than the actual formal dimension on the Gromov-Taubes side, $(\sum_{i \leq k}(dim_{\mathbf{R}}B + e_i^2 - e_i \cdot c_1(\mathbf{K})) - (k-1)dim_{\mathbf{R}}B$.

Motivated from Mcduff's proposal [Mc] and Taubes' theorem 1, let us denote P as the index subset $P \subset \{1, 2, \cdots, k\}$ such that $e_i \cdot C < 0, i \in P$, with $e_i^2 < 0$. Then we can always regroup the decomposition of C as

$$C = F + E, F = \sum_{i \notin P} m_i e_i; E = \sum_{j \in P} m_j e_j.$$

Namely, one can view F as a whole without going into the details of the decomposition of the class F.

Then the previous expression can be expanded easily into the following

$$dim_{\mathbf{R}}B + 2d_{GT}(C) = F^2 - c_1(\mathbf{K}) \cdot F + 2F \cdot E + E^2 - c_1(\mathbf{K}) \cdot E + dim_{\mathbf{B}}B.$$

Then we rewrite the term $F \cdot E$ term into $F \cdot E = (C-E) \cdot E$, then we have

$$dim_{\mathbf{R}}B + 2d_{GT}(C) = F^2 - c_1(\mathbf{K}) \cdot F + 2(C-E) \cdot E + E^2 - c_1(\mathbf{K}) \cdot E + dim_{\mathbf{R}}B.$$

Let us collect $F^2 - c_1(\mathbf{K}) \cdot F + dim_{\mathbf{R}}B + \sum_{i \in P}(e_i^2 - c_1(\mathbf{K}) \cdot e_i)$ into a single term. It is the family dimension of F, along with the all the $e_i, i \in P$. The sum of the left-over terms, called the family dimension discrepancy, has the following form

$$\Delta_C(E) = 2C \cdot E - E^2 - c_1(\mathbf{K}) \cdot E - \sum_{i \in P}(e_i^2 - c_1(\mathbf{K}) \cdot e_i).$$

To study how do the multiplicities $m_i, i \in P$ in $E = \sum m_i E_i$ affect the family dimension, we have to introduce some combinatorial language. combinatorial language.

Definition 5 *Let C be a monodromy invariant fiberwise cohomology class of $\pi : \mathcal{X} \mapsto B$. Given a point $b \in B$, Let $e_i, i \in P$ be all the classes represented by irreducible pseudo-holomorphic curves over b with $C \cdot e_i < 0, e_i^2 < 0$. Then define the exceptional cone of C over b to be $\mathcal{EC}_b(C) = \sum_i \mathbf{R}^{\geq 0} e_i \in H^2(\pi^{-1}(b), \mathbf{R})$.*

As E is written as$\sum m_i e_i$ with $m_i \geq 0$. We can view E as an element in the cone $\mathcal{EC}_b(C)$ generated by these e_i.

Given such a cone $\mathcal{EC}_b(C) \in H^2(\pi^{-1}(b), \mathbf{R})$ with an indefinite intersection form, it is possible to define the dual cone $\mathcal{EC}_b^*(C)$ to be the elements in $H^2(X, \mathbf{R})$ which have non-negative intersection pairings with elements in $\mathcal{EC}_b(C)$. As $\mathcal{EC}_b(C)$ is usually not a top dimensional cone in H^2, the dual cone $\mathcal{EC}_b^*(C)$ usually is the direct sum of a vector space cone with a reduced dual cone in the minimal subspace containing $\mathcal{EC}_b(C)$.

As a preparation, we want to prove some simple lemma characterizing the cone $\mathcal{EC}_b(C)$.

Let us review some definitions.

Definition 6 *Let $\mathcal{EC}_b(C)$ be an exceptional cone as described before. Then $\mathcal{EC}_b(C)$ is said to be admissible over b if $(C - \mathcal{EC}_b(C)) \cap \mathcal{EC}_b(C)^*$ contains at least one lattice point.*

The following proposition clarifies the relation between the intersection form on $H^2(\pi^{-1}(b), \mathbf{R})$ with the admissible cone $\mathcal{EC}_b(C)$.

Proposition 3 *Suppose that the exceptional cone over b, $\mathcal{EC}_b(C)$, is admissible. Then the restriction of the intersection quadratic form on the cone $\mathcal{EC}_b(C)$ is negative definite.*

This proposition implies that the term $-E^2$ in $\Delta_C(E)$ always contributes positively when $E \in \mathcal{EC}_b(C)$.

Notice that we do not claim that the intersection form is negative definite on the whole minimal vector space in H^2 containing $\mathcal{EC}_b(C)$. If the fibration $\pi : \mathcal{X} \mapsto B$ is algebraic. Then all the e_i are of type $(1,1)$. Recall that Hodge index theorem asserts the intersection form has only one positive eigenvalue. In this case this proposition asserts that the exceptional cone is disjoint to the forward and backward light cones.

The proposition is a generalization of Mcduff's proposal.

Remark 1 *When $B = pt$, the only exceptional curves satisfying (ii). are -1 curves. Suppose $e_i, e_j \in \mathcal{EC}_{pt}(C)$ are two different -1 curves. Then the proposition implies that $e_i + e_j \in \mathcal{EC}_{pt}(C)$ is of negative square. In other words, $(e_i + e_j)^2 = -2 + 2e_ie_j < 0$. Then $e_i \cdot e_j$ must be 0. This recovers Mcduff's proposal that all -1 curves $e_i, e_i \cdot C < 0$ must be perpendicular to each other.*

Proof of the Proposition: Set $|P| = n$. As $\mathcal{EC}_b(C)$ is admissible, then there must be some tuple $(m_1, m_2, \cdots, m_n) \in \mathbf{N}^n$ such that $(C - \sum_{1 \leq i \leq n} m_i e_i) \cdot e_j \geq 0$ for all $j \in P$. For simplicity let us re-scale and use the $\underline{e_i} = m_i e_i$ as the new generators. Notice that it is related to the old one by a positive scaling. Then we have the following inequality

$$(\underline{e}_1 + \underline{e}_2 + \cdots + \underline{e}_n) \cdot \underline{e}_i \leq C \cdot \underline{e}_i < 0,$$

for all $i \in P$. Now we use that $\underline{e}_i \cdot \underline{e}_j \geq 0$ for $i \neq j$.

Then we must have more inequalities of the similar type. Let $S \subset P$ be any nonempty subset of the index set P. Then

$$(\sum_{i \in S} \underline{e}_i) \cdot \underline{e}_j < -(\sum_{i \in P-S} \underline{e_i} \cdot \underline{e_j}) < 0, j \in S.$$

Lemma 1 *Given any two index subsets $A, B \subset P$ such that one includes the other. Then it follows that $\{\sum_{i \in A} \underline{e}_i\} \cdot \{\sum_{j \in B} \underline{e}_j\} < 0$.*

Proof of the Lemma: By symmetry we may assume $A \supset B$. Then we take $A = S$ in the above discussion and we have

$$(\sum_{i \in A} \underline{e}_i) \cdot \underline{e}_j < 0, j \in A.$$

We may choose $j \in B \subset A$ and sum up all these inequalities for j running through B, we get

$$(\sum_{i \in A} \underline{e}_i) \cdot (\sum_{j \in B} \underline{e}_j) < 0.$$

□

Now we are ready to prove the statement in the proposition. Let x be any element in the cone $\mathcal{EC}_b(C)$. Then it can be written as $\sum_{i\in P} c_i \underline{e}_i$ with $c_i \in \mathbf{R}+$.

For convenience let us rearrange the indexes in P in such a way that the coefficients c_i are monotonically decreasing for increasing i. After such a permutation of indexes, let us consider elements of the form $f_a = \sum_{a\geq j\geq 1} \underline{e}_j$. Then the same element x can be written alternatively as

$$E = \sum_{l\leq n} (c_l - c_{l+1}) f_l,$$

where we have set $c_{n+1} = 0$. Therefore E is written as an effective (because $c_l - c_{l+1} \geq 0$ expression over the new generators $f_l s$.

On the other hand, the index sets involve in defining f_l form a monotonic chain. They are of the form $\{1\} \subset \{1,2\} \subset \{1,2,3\} \cdots$.

Therefore $f_l \cdot f_{l'} < 0$ for all pairs of (l, l'). Then it is easy to see that

$$E \cdot E = \sum_{l,l'\leq n} (c_l - c_{l+1})(c_{l'} - c_{l'+1}) f_l \cdot f_{l'} < 0.$$

The equality can hold only when $c_l - c_{l+1} = 0$ for all l. As c_{n+1} is defined to be zero, all the c_l must vanish. In other words, E is the zero element of the cone $\mathcal{EC}_b(C)$. □

Proposition 4 *The cone $\mathcal{EC}_b(C)$ is simplicial. Namely, the generators $e_i, i \in P$ are all linear independent.*

Proof of prop. 4 Suppose that e_i are linearly dependent and there is a linear equation $\sum a_i e_i = 0$. We can move all the terms with negative a_i to the right hand side and rewrite the equation as

$$\sum_{i\in B} b_i e_i = \sum_{j\in B'} b_j e_j, B \cap B' = \emptyset.$$

This tells us that a single element in $\mathcal{EC}_b(C)$ has more than one expression in terms of the generators e_i.

By prop. 3, we can calculate

$$0 > (\sum_{i\in B} b_i e_i) \cdot (\sum_{i\in B} b_i e_i) = (\sum_{j\in B'} b_j e_j) \cdot (\sum_{i\in B} b_i e_i) \geq 0,$$

as $B \cap B' = \emptyset$. Contradiction! □

The proposition 3 implies that $\mathcal{EC}_b(C) \cap \mathcal{EC}_b^*(C) = \{0\}$. Namely the dual cone is completely disjoint with the original exceptional cone. On the other hand it is easy to see that

$(C - \mathcal{EC}_b(C)) \cap \mathcal{EC}_b(C)^* \neq \emptyset$ if and only if

$$(\mathcal{EC}_b(C) - C) \cap (-\mathcal{EC}_b^*(C)) \neq \emptyset$$

if and only if

$$\mathcal{EC}_b(C) \cap (C - \mathcal{EC}^*_b(C)) \neq \emptyset$$

We will study the subset of lattice points in $\mathcal{EC}_b(C) \cap (C - \mathcal{EC}^*_b(C))$ closely.

Definition 7 *Define the discrete set $\Lambda_b(C)$ to be the the lattice points in $\mathcal{EC}_b(C) \cap (C - \mathcal{EC}^*_b(C))$.*

The elements in $\Lambda_b(C)$ are the expression $\sum_{i \in P} m_i e_i$, $m_i \in \mathbf{Z}^{\geq 0}$, such that $(C - \sum_{i \in P} m_i e_i) \cdot e_j \geq 0, j \in P$.

Let us list the basic properties of the set $\mathcal{EC}_b(C) \cap (C - \mathcal{EC}^*_b(C))$ in the following simple proposition,

Proposition 5 *Let $E(C)$ be the intersection $\mathcal{EC}_b(C) \cap (C - \mathcal{EC}_b(C)^*)$ of two different translated cones. As a proper subset of an affine space, $E(C)$ is convex as well as unbounded. In fact, given any point z in $E(C)$, we consider the ray $tz, t \geq 1$, then this ray is entirely contained in $E(C)$.*

Proof: The statement regarding convexity is trivial. Suppose $z \in E(C)$. Then $(C - z) \cdot e_i > 0$ for all $i \in P$. Then $z \cdot e_i < C \cdot e_i < 0$ and we can rewrite $(C - tz) = (C - z) - (t - 1)z$ and $(C - tz) \cdot e_i = (C - z) \cdot e_i - (t - 1)z \cdot e_i > 0$ for all $i \in P$. Then $tz \in E(C)$. $\Box$

Because

$$2d_{GT}(C) + dim_{\mathbf{R}}B = F^2 - c_1(\mathbf{K}) \cdot F + dim_{\mathbf{R}}B + \sum_{i \in P}(e_i^2 - c_1(\mathbf{K}) \cdot e_i) + \Delta_C(E),$$

we are interested at knowing when does $\Delta_C(E)$ take non-positive values.

Definition 8 *An lattice element E in $\Lambda_b(C) \subset \mathcal{EC}_b(C)$ is said to be allowable with respect to C if the function value $\Delta_C(E)$ is non-positive.*

The corresponding decomposition $(C - E, E = \sum m_i e_i)$ is said to be an allowable decomposition.

Let us look at the formula

$$\Delta_C(E) = 2C \cdot E - E^2 - c_1(\mathbf{K}) \cdot E - \sum_{i \in P}(e_i^2 - c_1(\mathbf{K}) \cdot e_i).$$

If we view E as a moving variable in $\Lambda_b(C)$ and take an element z in $\Lambda_b(C)$, then $nz, n \geq 1$ form a sequence of lattice points in $\Lambda_b(C)$. Using the fact that the term $E \cdot E$ is negative definite, we find that the leading quadratic term in n, $n^2 z \cdot z$ is always positive. Thus it dominates all the other linear or constant terms for the large enough n.

Thus, even though the set $\Lambda_b(C)$ is unbounded, the function $\Delta_C(E)$ is bounded below in the un-compact end. As a result, the function $\Delta_C(E)$: $\Lambda_b(C) \mapsto \mathbf{Z}$ must attain its absolute minimum somewhere.

Before discussing the geometric meaning, let us point out that the locations of the minimums only depend on the numerical data and is universally independent to the geometric data of the fiber $\pi^{-1}(b)$ itself.

In principle, the elements nz, $n \mapsto \infty$ in the rays will not give rise to effective decomposition $C = (C - nz) + nz$ (because $(C - nz) \cdot \omega_{\mathcal{X}/B} \mapsto -\infty$) for large enough n. It gives an alternative reason to discard the non-compact end of $\Lambda_b(C)$ as one discusses the function $\Delta_C(E)$. However, the non-effectiveness (not being representable by pseudo-holomorphic curves) depends on the geometric data of the fiber and is not as numerical as the dimension constraint.

In principle, the location of the actual minimums relies on the data of the various intersection numbers $C \cdot e_i, e_i^2, e_i \cdot e_j$. To characterize these lattice points geometrically we have to give them a suitable interpretation.

To study the geometric meaning of the minimums of $\Delta_C(E)$, let us start with the $|P| = 1$ case first. Let us assume that $\mathcal{EC}_b(C)$ is a one dimensional cone generated by a single e with $e^2 < 0$. One suppose that e is represented by an irreducible (pseudo)-holomorphic curve above $b \in B$.

Therefore we write

$$e^2 + c_1(\mathbf{K}) \cdot e = 2g - 2,$$

where g is the arithmetic genus of e and is usually bigger than the geometric genus of e unless the curve dual to e is smooth. Any lattice point in $\mathcal{EC}_b(C)$ can be written as $m \cdot e, m \in \mathbf{N}$. Then the function $\Delta_C(E) : \{\mathbf{N}e\} \mapsto \mathbf{Z}$ can be simplified to

$$\Delta_C(m) = 2m(e \cdot C) - m^2 e^2 - m(2g - 2 - e^2) - (2e^2 - 2g - 2).$$

It is easy to find the value $m_0 \in \mathbf{N}$ for which $\Delta_C(m)$ is minimized. We replace m by a real variable x and the minimum is achieved when the derivative is zero. In other words, one has

$$2e \cdot C - 2x \cdot e^2 - 2g + 2 + e^2 = 0.$$

The real number x satisfying this equation is rational. To see the geometric meaning of the solution, let us consider the expression $e \cdot C - g$, which is always a negative number.

From simple arithmetic means, it is always possible to re-write $eC - g$ as $l \cdot e^2 + r$ with l, r non-negative, with r being the remainder ,$0 \leq r < -e^2$. It is easy to see that $m = l$ is the closest lattice point to the actual minimum of the function $\Delta_C(E)$. It is due to the fact that $f(x) = e \cdot C - xe^2 - g + 1 + \frac{e^2}{2}$ has the property $f(x+1) = f(x) - e^2$ and $\frac{e^2}{2} < f(l) = r + 1 + \frac{e^2}{2} \leq \frac{-e^2}{2}$. Therefore $l \cdot e$ is the unique lattice point in this range. Moreover it has the crucial property that the function $d(m) = \Delta_C(me)$ is monotonically decreasing for $m \geq m_{cri} = l$.

In the actual application of the curve counting scheme in the enumerative application, the $g = 0$ case is the most interesting situation.

The significant simple property of the function $\Delta_C(E)$ in the one dimensional case will be used frequently later. Let us summarize it as a lemma.

Lemma 2 *(Moving lemma) Suppose that all the classes e_i satisfy $g(e_i) = 0$. Then the function $\Delta_C(E) : \Lambda_b(C) \mapsto \mathbf{Z}$ is monotonically decreasing if one moves from E to $E + e_i$, for all i.*

Proof: Suppose that e_i has been fixed in our discussion. First we notice that E can be rewritten as $E = E_0 + m_i e_i$ where $E_0 = \sum_{j \neq i} m_j e_j$.

Then we compare $\Delta_C(E)$ and $\Delta_{C-E_0}(m_i e_i)$ and see

$$\Delta_C(E) - \Delta_{C-E_0}(m_i e_i) = 2C \cdot E - E^2 - c_1(\mathbf{K}) \cdot E - 2(C - E_0) \cdot (m_i e_i) + (m_i e_i)^2 + c_1(\mathbf{K}) \cdot (m_i e_i)$$

$$= 2E_0 \cdot (m_i e_i) - (E_0 + m_i e_i)^2 + (m_i e_i)^2 - c_1(\mathbf{K}) \cdot E_0 = -E_0^2 + c_1(\mathbf{K}) \cdot E_0,$$

is a constant independent of m_i. To show that $\Delta_C(E)$ is monotonically decreasing, it suffices to show that $\Delta_{C-E_0}(m_i e_i)$ is monotonically decreasing in m_i for $m_i e_i$ in $\Lambda_b(C - E_0)$. This reduces the problem to the one dimensional case, which has been discussed earlier.

□

Remark 2 *It should be emphasized that in the single e with $g(e) = 0$ case, the integer l making $\Delta_C(E)$ minimum is the first positive integer m such that $(C - me)$ lies in the dual cone $\mathcal{EC}_b^*(C)$. If C is represented by (pseudo) holomorphic curves, then the curve dual to C can never be irreducible. As $C \cdot e < 0$, C must split off as as a curve dual to $C - me$ and one dual to me with multiplicity m. Symbolically we may write $C = (C - me) + me$ as a decomposition of the cohomology classes to represent the splitting of curves. The only general requirement upon m is that $C - me$ and e exist simultaneously as pseudo-holomorphic curves above the fiber of b. Thus $(C - me) \cdot e \geq 0$ and it follows that $m \geq \frac{C \cdot e}{e^2} \geq l$. On the other hand, there is no a priori constraint about m other than the previous inequality.*

On the other hand, for rational e, with $e^2 + c_1(\mathbf{K}) \cdot e = -2$, the minimal choice $m = l$ to make $(C - me) \cdot e \geq 0$ has the additional nice property that it makes the dimension discrepancy function $\Delta_C(E)$ minimized. In other words, the curve in the class C tends to split off(bubbling off) a certain curves dual to m multiple of e such that $C - me$ has nonnegative intersection with e. The minimum amount $m = l$ also makes the family moduli space dimension $dim_{\mathbf{R}} B + d_{GT}(e) + d_{GT}(C - me)$ largest.

However this nice topological interpretation does not hold for higher genera case. In fact, if the arithmetic genus g is larger than 0, then the role of $e \cdot C$ is replaced by $e \cdot C - g$ and it is $e \cdot C - g$ instead of $e \cdot C$ which is represented as the form $le^2 + r$. Thus the minimum of $\Delta_C(E)$ usually takes place for a larger integer than what the naive topological constraint predicts. In other words, the largest expected family dimension happens for the multiplicity m_{cri} larger than the topological constraint by the integer $[\frac{g}{-e^2}]$ or $[\frac{g}{-e^2}] + 1$.

It is desirable to identify where can the minimum values of $\Delta_C(E)$ occur in $\Lambda_b(C)$.

Let us consider the translated cones $C - \mathcal{EC}_b(C)^* + e_i$, the translate of the shifted dual cone $C - \mathcal{EC}_b^*(C)$ by the canonical basis elements e_i. Then we have the translated version of the lattice points $\Lambda_b(C)$ in the corresponding convex set $\mathcal{EC}_b(C) \cap C - \mathcal{EC}_b^*(C)$, denoted by $\Lambda_b(C)_i$. Next we consider the set

$$M_b(C) = \Lambda_b(C) - \cup_i(\Lambda_b(C)_i).$$

The following proposition asserts that $M_b(C)$ is a non-empty finite set.

Proposition 6 *Let $M(C)$ be the set as defined above, then the set $M(C)$ is non-empty and finite.*

Proof: Suppose that $M_b(C)$ is empty, then it follows that $\Lambda_b(C) \subset \cup_i(\Lambda_b(C)_i)$. Let us pick an arbitrary element $\lambda^{(0)} = \lambda$ in the left hand side. Then it must be in one of the $\Lambda_b(C)_i$. In other words, $C - (\lambda - e_i)$ also lies in $\mathcal{C}_E^*$. However this implies that $\lambda^{(1)} = \lambda - e_i$ is also in $\Lambda(C)$.

From here we conclude that for any element $\lambda^{(0)} = \lambda$ in $\Lambda(C)$, there must be some i such that $\lambda^{(2)} = \lambda - e_i$ still lies in the set $\Lambda_b(C)$. Then by induction, one may use $\lambda - e_i$ instead of λ and conclude $\lambda^{(2)} = (\lambda - e_i) - e_j$ is still in $\Lambda_b(C)$. However it is impossible as it implies that given an element $\lambda^{(0)} = \lambda$, one can indefinitely shifts it backward and get another lattice point. Because there are only a finite number e_i, we must get a lattice point $\lambda^h \in \Lambda_b(C) \subset \mathcal{EC}_b(C)$, $h \gg 0$ with at least a negative coordinate entry with respect to some $e_i, i \in P$. Contradiction! Thus the existence of the lattice point in $M_b(C)$ has been derived.

To show that $M_b(C)$ is finite, firstly we notice that $M_b(C)$ is a discrete subset in $H^2(\pi^{-1}(b), \mathbf{R})$.

Lemma 3

The set $\overline{\mathcal{EC}_b(C) \cap (C - \mathcal{EC}_b^*(C)) \cap \left(\cup_i(C - \mathcal{EC}_b^*(C) + e_i)\right)^c}$ is compact in $H^2(\pi^{-1}(b), \mathbf{R})$.

Proof of the lemma: Apparently the set is closed, we only need to check it is a bounded subset of $\mathcal{EC}_b(C) \subset H^2(\pi^{-1}(b), \mathbf{R})$.

Suppose that $E \in \overline{\mathcal{EC}_b(C) \cap (C - \mathcal{EC}_b^*(C)) \cap \left(\cup_i(C - \mathcal{EC}_b^*(C) + e_i)\right)^c}$, then $E \in \mathcal{EC}_b(C) \cap (C - \mathcal{EC}_b^*(C))$ but E is not in the interior of $\mathcal{EC}_b(C) \cap (C - \mathcal{EC}_b^*(C) + e_i)$ for all $i \in P$.

Thus, $(C - E) \cdot e_i \geq 0, i \in P$ but $(C - E + e_j) \cdot e_i \leq 0, i, j \in P$.

This implies that the values $E \cdot e_i, i \in P$ are bounded by

$$min_{j \in P}(C + e_j) \cdot e_i \leq E \cdot e_i \leq C \cdot e_i, i \in P.$$

This implies that the pairing functionals $e_i \cdot \circ$ take bounded values on E. On the other hand, $e_i, i \in P$ form a basis of the minimal vector space containing $\mathcal{EC}_b(C)$, the pairing functional $e_i \cdot \circ$ is a linear coordinate system. As E has bounded coordinates, such E forms a bounded subset in $\mathcal{EC}_b(C)$. $\square$

From the fact that $\overline{\mathcal{EC}_b(C)\cap(C-\mathcal{EC}_b^*(C))\cap(\cup_i(C-\mathcal{EC}_b^*(C)+e_i))^c}$ is compact and $M_b(C)$ is a discrete subset, it follows that $M_b(C)$ has to be a finite set. $\Box$

Usually it is not clear whether the lattice points in the finite set $M_b(C)$ are unique or not. The following proposition clarifies the significance of the lattice points in $M_b(C)$.

Proposition 7 *Let $E=\sum_i m_ie_i, m_i\in\mathbf{N}$ be in $\Lambda_b(C)\subset\mathcal{EC}_b(C)\cap(C-\mathcal{EC}_b^*(C))$. Then there exists at least one element z in $M_b(C)$ and a finite sequence of elements $z_p\in\Lambda_b(C)$, $1\leq p\leq N$ such that*

(i). $z_1=z$, $z_N=E$.
(ii). If $p\neq 1$, then $z_p-z_{p-1}=e_i$ for some $1\leq i\leq |P|$.

Proof of the proposition: $\Box$ Given an element $E\in\Lambda_b(C)$, if E also lies in $M_b(C)$, we are done. We set $N=1$ and $z_1=E=z\in M_b(C)$.

Otherwise if $x_0=E\notin M_b(C)$, then $E\in\Lambda_b(C)_i$ for some $1\leq i\leq|P|$. In other words, there exists an $x_1\in\Lambda_b(C)$ such that $x_0=E=x_1+e_i$ for some $1\leq i\leq|P|$.

Take $x_1\in\Lambda_b(C)$ and repeat the argument, either one gets $x_2\in\Lambda_b(C)$ such that $x_1=x_2+e_i$ for some $1\leq i\leq|P|$ or $x_2\in M_b(C)$. By induction, one may get a sequence of points $x_n\in\Lambda_b(C)$. One argues that for a large enough x_n, we must have $x_n\in M_b(C)$, or the induction process never stops and one gets an infinite sequence $x_n\in\Lambda_b(C), n\in\mathbf{N}$. Because there are a finite number of $e_i, 1\leq i\leq|P|$, such an $x_n=\sum_{1\leq i\leq|P|}\underline{m}_ie_i$ must has a negative entry $m_j<0$ for some $j\leq|P|$ and falls out of $\mathcal{EC}_b(C)$. Contradiction!

Then we take $z_1=z=x_n\in M_b(C)$ and rename the sequence $x_p, 1\leq p\leq n$ by $z_q=x_{n+1-q}, q\leq n+1=N$.

Such a finite sequence of lattice elements satisfies
(i). $z_1=z\in M_b(C)$, $z_N=x_0=E\in\Lambda_b(C)$. $z_q\in\Lambda_b(C)$
(ii). For $p>1$, $z_p-z_{p-1}=e_i$ for some $i\leq|P|$.

We are done. $\Box$

We assume that $g(e_i)=0$ for all $1\leq i\leq|P|$ in the following remark.

Remark 3 *Suppose that C is represented by a pseudo-holomorphic curve over the point $b\in B$, then one argues that the curve must contain irreducible curves dual to the various $e_i, i\leq|P|$. It is because e_i is known to be irreducible and pseudo-holomorphic above $b\in B$, then $C\cdot e_i\geq 0$ if all the irreducible components of the curve dual to C are distinct from the one dual to e_i.*

Thus, one may write $C=(C-\sum_i m_ie_i)+(\sum_i m_ie_i)$ with $m_i\in\mathbf{N}$, where $(C-\sum_i m_ie_i)$ is dual to a curve disjoint from all the exceptional curves dual to e_i. Then we must have $(C-\sum_i m_ie_i)\cdot e_j\geq 0$ for all $j\leq|P|$. If we take $E=\sum_i m_iE_i\in\mathcal{EC}_b(C)$, we must have $(C-E)\in\mathcal{EC}_b^*(C)$. In other words, $E\in\Lambda_b(C)\subset\mathcal{EC}_b(C)\cap\big(C-\mathcal{EC}_b^*(C)\big)$.

We have use an E to resemble the pattern C splits into different multiples of $e_i, i \leq |P|$.

If one imagines a pseudo-holomorphic curve dual to $C - E_0$ splits into a curve dual to $C - E_0 - e_i$ and a curve dual to e_i as a degeneration process (known as the bubbling off phenomenon in symplectic geometry), then the lattice move $E_0 \mapsto E_0 + e_i$ in $\Lambda_b(C)$ is equivalent to "bubbling off" an unit of e_i from C.

The path from z to E by a finite sequence of lattice moves as in the condition (ii) of prop. 7 indicates that the curve dual to the sum of $(C - E)$ and E can be viewed formally as degenerated from a curve dual to $C - z$ and z by a finite number of bubbling offs of $e_i, i \leq |P|$.

(a). In other words, if we consider the moduli space (and its bubbling off) of curves dual to C splitting into one dual to $C - z$ and a combination of multiple coverings of exceptional curves dual to z, it contains the given curve dual to the sum of $C - E$ and E.

(b). Given any pseudo-holomorphic curve dual to C, one may assign an unique $E \in \Lambda_b(C)$ to it. By prop. 7, one may associate the point $z \in M_b(C)$ to E with the properties,

(i). The decomposition C into $(C - z) + z$ has the highest expected family Gromov-Taubes dimension among all the decompositions into distinct pseudo-holomorphic curves related by the elementary moves.

(ii). The curve dual to $(C - E) + E$ can be thought as a degenerated version of curves dual to $(C - z) + z$ by bubbling off a few rational exceptional curves dual to the e_i.

When one applies the family switching formula of rational exceptional curves, one is able to rewrite some mixed family invariant of $C - \sum e_i$ over a locus over which all the exceptional curves in $e_i, i \leq |P|$ co-exist, (with all the multiplicities $m_i \equiv 1$) in terms of the family invariants of $C - z$, $z = \sum_{1 \leq i \leq |P|} n_i e_i \in M_b(C)$. Combine with remark 3, this tells us that formally the mixed family invariant can be interpreted as the counting of curves in the class $C - \sum_{1 \leq i \leq |P|} n_i e_i$.

Returning to the pointwise discussion over $b \in B$, any two different lattice points within $M_b(C)$ cannot be related by each other by effective translation. i.e. shifting from one to another by $\sum_i c_i e_i, c_i \leq 0$. In reality the geometric meaning of shifting toward the right means degeneration or bubbling off phenomena(in the case $g(e_i) = 0$). In this way, we may give the lattice points in $\Lambda_b(C)$ a partial ordering and different elements in $M_b(C)$ are the maximum elements (not necessarily the greatest element) of the partial ordering.

Definition 9 *Let λ_1 and λ_2 be two different lattice elements in $\Lambda_b(C)$. We say that λ_1 is greater than λ_2, denoted by $\lambda_1 \sqsupset \lambda_2$ if $lambda_2$ can be gotten from λ_1 by a finite sequence of effective lattice translations, i.e. there exists $\lambda_2 = z_n, z_{n-1}, z_{n-2}, \cdots, \lambda_1 = z_1 \in \Lambda_b(C)$ such that $z_p - z_{p-1} = n_p e_{i_p}$ for some $n_p \in \mathbf{N}$, $1 \leq i_p \leq |P|$. It is apparent that this relation $\sqsupset$ is transitive.*

We say that an element λ is a maximal element if there is no other element in $\Lambda_b(C)$ which is greater than it under the partial ordering $\sqsupset$.

Then the set $M_b(C)$ consists of the maximum elements in $\Lambda_b(C)$. This justifies the notation $M_b(C)$ and its dependence on the class C.

From the previous lemma 2 one finds that if $\lambda_1 \sqsupset \lambda_2$, then $\Delta_C(\lambda_1) < \Delta_C(\lambda_2)$. Therefore, in the rational case, $M_b(C)$ also represents the lattices points of $\Lambda_b(C)$ whose $\Delta_C(E)$ values are smallest. Even though the values of $\Delta_C(E)$ are different for the different lattice points, it is no use to compare their value as they are not linked to each other by effective lattice shifting. On the other hand, our discussion does not rule out the possibility that by different effective shiftings one can reach from two different maximum elements $\in M_b(C)$ to the same lattice point $\lambda \in \Lambda_b(C)$. As effective shiftings correspond to pseudo-holomorphic curve degenerations, the single decomposition $(C-\lambda, \lambda)$, $\lambda \in \Lambda_b(C)$ can possibly be degenerated from two distinct maximal decompositions $(C-\lambda_1, \lambda_1), (C-\lambda_2, \lambda_2)$, $\lambda_1, \lambda_2 \in M_b(C)$, $\lambda_1 \sqsupset \lambda, \lambda_2 \sqsupset \lambda$.

In the latter part of the paper, we would like to discuss how does the degeneration process affect the relative obstruction bundles. The family switching formula assigns a relative obstruction bundle to a degeneration of decompositions such that $\Delta_C(E) - \Delta_C(E + \sum n_i e_i)$ is directly related to the rank of the bundle. It turns out that not only on the numerical level do the partial ordering organizes the lattice points of $\Lambda_b(C)$ in a nice way, but they also co-relate the relative obstruction bundles. This will be the main focus of the next section.

2.1 The Irrational, $g(e_i) > 0$, Cases

In the previous discussion, we have focused upon the $g(e_i) = 0$ case. Let us consider the general situation that arithmetic genera $g(e_i), i \leq |P|$ may be non-zero. As we discuss earlier (see page 15), the appearance of the genus term shifts the minimum value of $\Delta_C(E)$. This happens even when the exceptional effective cone is of one dimension.

Let us make this simplifying assumption temporally. Once we have a closer look at the formula $-x \cdot e^2 + e \cdot C - g + 1 + e^2/2$, we may collect $e\dot{C} - g + 1$ into a single term with a particular topological meaning. Imagining that e is represented by a smooth pseudo-holomorphic curve in $\pi^{-1}(b)$. Suppose C is the first Chern class of a holomorphic line bundle $\mathbf{F}$ on the Riemann surface $\Sigma \subset \pi^{-1}(b)$. Then the expression $e \cdot C - g + 1 = \int_\Sigma c_1(\mathbf{F}) - g(\Sigma) + 1$ resembles the holomorphic Euler number of the holomorphic line bundle $\mathbf{F}$. The appearance of the special number indicates that the tangent-obstruction complex would contain a term associated to $H^0(\Sigma, \mathbf{F}) - H^1(\Sigma, \mathbf{F})$, $e = PD[\Sigma]$.

In general, consider the intersection matrix $I_{i,j} = e_i \cdot e_j$, $1 \leq i, j \leq |P|$. Let us take the dual basis e_i^* with respect to e_i such that $e_i \cdot e_j^* = I_{i,j}\delta_{i,j}$, and the normalized dual basis $\frac{e_j^*}{e_j \cdot e_j} = \hat{e}_j$. Then we have $e_i \cdot \hat{e}_j = \delta_{i,j}$ for $1 \leq i, j \leq |P|$.

As the classes C and $c_1(\mathbf{K}_{\mathcal{X}/B})$ may not lie in the cone $\mathcal{EC}_b(C)$, we project them orthogonally into the subspace $\mathcal{EC}_b(C) \otimes \mathbf{R}$. Denote $\hat{C} := \sum_{1 \leq i \leq |P|} (C \cdot e_i)\hat{e}_i$ and $\hat{K} := \sum_{1 \leq i \leq |P|} (c_1(\mathbf{K}_{\mathcal{X}/B}) \cdot e_i)\hat{e}_i$, then we must have

$$c_1(\mathbf{K}_{\mathcal{X}/B})\cdot e_j = \hat{K}\cdot e_j, C\cdot e_j = \hat{C}\cdot e_j.$$

The $\hat{C}$ and $\hat{K}$ lie in the subspace $\mathcal{EC}_b(C)\otimes\mathbf{R}\subset H^2(M,\mathbf{R})$ and are the projection of C and $c_1(\mathbf{K}_{\mathcal{X}/B})$ to the subspace. These elements $\hat{C}$ and $\hat{K}$ are rational points of the minimal subspace $\mathcal{EC}_b(C)\otimes\mathbf{R}$.

Differentiating $\Delta_C(E)|_{E=\sum m_i e_i}$ with respect to the variable m_i, one derives that

$$-2E\cdot e_i + 2\hat{C}\cdot e_i - \hat{K}\cdot e_i = 0, 1\leq i\leq |P|.$$

Then one concludes that the rational point $E=-\hat{C}+\frac{1}{2}\hat{K}$

is the minimum of the quadratic function $\Delta_C(E)$. By using the adjunction equality one may rewrite

$$\hat{K}=\sum_i(2g(e_i)-2-e_i^2)\hat{e}_i,$$

$$\hat{C}-\frac{1}{2}\hat{K}=\sum_i(e_i\cdot C-g(e_i)+1+\frac{e_i^2}{2})\hat{e}_i.$$

If the arithmetic genera $g(e_i)$ change from zero to positive values, then we find the minimum of the function $\Delta_C(E)$ are shifted from their original value by the rational vector $-g(E):=-\sum_{1\leq i\leq |P|}g(e_i)\hat{e}_i$. It is easy to see that $g(E)\in\mathcal{EC}_b(C)\otimes\mathbf{R}$ is a rational element in $\mathcal{C}_E^*$; the dual cone $\mathcal{EC}_b^*(C)$. As we have proved (in prop. 3) that $\mathcal{EC}_b(C)\cap\mathcal{EC}_b^*(C)=\{0\}$, so the element $g(E)$ never lies in the original $\mathcal{EC}_b(C)$.

But it is not clear from the definition whether $-g(E)$ can be in $\mathcal{EC}_b(C)$ or not.

3 The Global Discussion and The Admissible Decomposition Classes

In the previous section, we have finished the pointwise discussion on the exceptional cones of C and the family dimensions. We would like to patch the local discussion over $b\in B$ together and introduce some additional structure on the base space B of the fibration $\pi:\mathcal{X}\mapsto B$.

Suppose that e is an exceptional class in the sense of definition 1. Suppose that at a given point $b\in B$ the class e has been represented by an irreducible holomorphic curve in $\mathcal{X}_b$, one may consider the locus $S_e\subset B$ over which the class e is effective. It is well known that S_e must be a compact subset of the compact set B.

By adjunction equality we have $e^2 + c_1(\mathbf{K}_{\mathcal{X}/B}) \cdot e = 2g_{arith}(e) - 2$. On the other hand, the Gromov theory predicts that the 'expected' dimension of the set S_e is $dim_{\mathbf{C}}B + \frac{e^2 - c_1(\mathbf{K}_{\mathcal{X}/B}) \cdot e}{2}$.

In order for the curve to be generic, i.e. expected dimension of $S_e \geq dim_{\mathbf{C}}B$, both inequalities

$$e^2 - c_1(\mathbf{K}_{\mathcal{X}/B}) \cdot e \geq 0, e^2 + c_1(\mathbf{K}_{\mathcal{X}/B}) \cdot e \geq -2$$

have to be satisfied. In particular, $e^2 = -1$ and e is represented by a genus zero smooth curve. Thus, we may conclude that the only generic exceptional curve within a family $\pi : \mathcal{X} \mapsto B$ has to be a smooth -1 curve.

For simplicity, let us assume that the arithmetic genus of the curve dual to e, $g_{arith}(e) = 0$, in the following discussion. Suppose that $e^2 = -n$, the irreducible rational curve representing e is called a $-n$ rational (exceptional) curve. According to the dimension formula, the expected complex dimension of S_e is $dim_{\mathbf{C}}B + \frac{d_{GT}(e)}{2} = dim_{\mathbf{C}}B + 1 - n$. This indicates when the self-intersection number $e \cdot e$ is more negative, the expected dimension of S_e would be much lower.

In general an irreducible holomorphic curve may degenerate and breaks into more than one irreducible component. Then there is a subset $S_e^{sm} \subset S_e$ over which the curve representing e is smooth and irreducible. The set $S_e - S_e^{sm}$ is the locus over which the curve representing e breaks into more than one component. In an ideal situation, S_e^{sm} is dense in S_e. Then $S_e - S_e^{sm}$ can be thought as the "boundary points" of S_e^{sm} collecting all the degenerated configurations of e.

When more than one e, say $e_1, e_2, \cdots, e_k$ are represented by holomorphic curves at the same $b \in B$, the set of all such b is nothing but $S_{e_1} \cap S_{e_2} \cdots \cap S_{e_k} = \cap_{i \leq k} S_{e_i}$.

In terms of intersection theory, the expected dimension of $\cap_{i \leq k} S_{e_i}$ is $dim_{\mathbf{C}}B + \frac{1}{2}\sum_{i \leq k} d_{GT}(e_i)$.

Let us list all the basic assumptions on a "perfect" set of C-exceptional classes.

Definition 10 *Let C be a $(1,1)$ class on $\mathcal{X}$ which restricts to non-trivial class on an algebraic fibration $\pi : \mathcal{X} \mapsto B$.*

A finite set of exceptional classes Q on $\pi : \mathcal{X} \mapsto B$ is said to be perfect if they satisfy the following list of basic assumptions.

Assumption 1 *(i). $e \cdot C < 0$.*
(ii). Either $S_e = \emptyset$ or S_e is a $dim_{\mathbf{C}}B + \frac{1}{2}d_{GT}(e)$ dimensional closed sub-variety of B.
(iii). The set S_e^{sm} is dense in S_e consisting smooth points of S_e.
(iv). If $\mathbf{e}$ is a holomorphic curve in $\mathcal{X}_b, b \in B$ representing e, then all the irreducible components of $\mathbf{e}$ represent exceptional classes. That is to say, none of the irreducible components has a non-negative self-intersection number. Moreover, if this exceptional class also satisfies (i). then it must also be in the original collection Q.

(v). Let $\mathbf{e}$ be a smooth irreducible holomorphic curve in $\mathcal{X}_b$ representing e and $b \in S_e^{sm}$. Then the composite morphism

$$\mathbf{T}_b B \mapsto H^1(\mathcal{X}_b, \Theta\mathcal{X}_b) \mapsto H^1(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}(\mathbf{e})) \mapsto H^1(\mathbf{e}, \mathcal{N}_{\mathbf{e}}\mathcal{X}_b)$$

induces an isomorphism $\mathbf{N}_{S_e^{sm}} B|_b \mapsto H^1(\mathbf{e}, \mathbf{N}_{\mathbf{e}}\mathcal{X}_b)$.
(vi). Suppose $e_1, e_2, \cdots, e_k$, $e_i \cdot e_j \geq 0$, $i \neq j$ are within the collection Q and are satisfying all (i).-(v)., then the locus of co-existence of $e_1, e_2, \cdots, e_k$, $\cap_{i\leq k} S_{e_i}$, is either empty or is a $dim_{\mathbf{C}} + \frac{1}{2}\sum_{i\leq k} d_{GT}(e_i)$ sub-variety in B.
(vii). The set $\cap_{i\leq k} S_{e_i}^{em}$ is dense in $\cap_{i\leq k} S_{e_i}$ containing the smooth points in it.

Under the condition (ii)-(iii) in the assumption 1, each S_e defines an algebraic cycle class $[S_e] \in \mathcal{A}_{dim_{\mathbf{C}} B + \frac{1}{2} d_{GT}(e)}(B)$ and can be used to define the algebraic family Seiberg-Witten invariant of e. The cycle class $[S_e]$ can be viewed as the moduli cycle of the exceptional class e.

Likewise, the locus of co-existence $\cap_{i\leq k} S_{e_i}$ of various $e_i, 1 \leq i \leq k$, also defines an algebraic cycle class $[\cap_{i\leq k} S_{e_i}]$, which is identical to the intersection cycle class $\cap_{i\leq k}[S_{e_i}] \in \mathcal{A}_{dim_{\mathbf{C}} B + \frac{1}{2} d_{GT}(e_i)}(B)$.

Remark 4 *In condition (v), the sheaf cohomology $H^1(\mathcal{X}_b, \Theta\mathcal{X}_b)$ parametrizes the infinitesimal Kodaira-Spencer deformations of complex structures on $\mathcal{X}_b$. Then $\mathbf{T}_b B \mapsto H^1(\mathcal{X}_b, \Theta\mathcal{X}_b)$ is the tautological map induced by the infinitesimal deformation of the family of algebraic surfaces $\mathcal{X} \mapsto B$. The morphism $H^1(\mathcal{X}_b, \Theta\mathcal{X}_b) \mapsto H^1(\mathbf{e}, \mathcal{N}_{\mathbf{e}}\mathcal{X}_b)$ is induced by the splitting $\Theta\mathcal{X}_b|_{\mathbf{e}} = \mathcal{T}_{\mathbf{e}} \oplus \mathcal{N}_{\mathbf{e}}\mathcal{X}_b$ along the holomorphic curve $\mathbf{e} \subset \mathcal{X}_b$.*

Let $b \in B$ and let $\mathcal{EC}_b(C)$ be as defined in definition 5. By proposition 4 the cone $\mathcal{EC}_b(C)$ is a simplicial cone generated by a collection of e_i represented by irreducible exceptional curves in $\mathcal{X}_b$.

Definition 11 *Define $\mathcal{EC}_b(C; Q)$ be the simplicial sub-cone of $\mathcal{EC}_b(C)$ generated by the elements $e_i \in Q$.*

As b moves on B, the cone $\mathcal{EC}_b(C; Q)$ often changes along with b. It makes sense to ask the following question,

Question: Describe the pattern of the variations of $\mathcal{EC}_b(C)$, $b \in B$, in terms of algebraic geometric datum on B.

Let $\mathcal{C}$ be a simplicial cone generated by elements in Q.

Definition 12 *Define $S_{\mathcal{C}} \subset B$ to be the set of all $b \in B$ such that $\mathcal{EC}_b(C; Q) \equiv \mathcal{C}$.*

For certain $\mathcal{C}$ there can be no such $b \in B$ and $S_{\mathcal{C}}$ is empty. We are only interested at those $\mathcal{C}$ with a non-trivial $S_{\mathcal{C}}$ and consider the pair $(S_{\mathcal{C}}, \mathcal{C})$.

Proposition 8 *Let Q be a perfect finite set of exceptional classes. Let $(S_{\mathcal{C}}, \mathcal{C})$ be a pair with $S_{\mathcal{C}} \neq \emptyset$. Then $S_{\mathcal{C}}$ is a locally closed subset of B and $S_{\mathcal{C}} \subset \overline{S_{\mathcal{C}}}$ consists of smooth points of $\overline{S_{\mathcal{C}}}$.*

Proof of proposition 8: Suppose that $\mathcal{C}$ is generated by $e_i \in Q, 1 \leq i \leq k$. Because Q is perfect, by assumption 1 (ii)., (iii)., (vi)., (vii)., $\cap_{1\leq i\leq k} S_{e_i}$ is a $dim_{\mathbf{C}} B + \frac{1}{2}\sum_{i\leq k} d_{GT}(e_i)$ dimensional sub-variety in B.

By our definition of $\mathcal{EC}_b(C; Q)$ as a subcone of $\mathcal{EC}_b(C)$, each $e_i \in \mathcal{C} \equiv \mathcal{EC}_b(C; Q)$ has to be represented by a smooth and irreducible exceptional curve above b. Thus, $S_{\mathcal{C}} \subset \cap_{1\leq i\leq k} S^{sm}_{e_i}$. The smoothness of $S_{\mathcal{C}}$ follows from condition (vii) of assumption 1.

We plan to argue that $S_{\mathcal{C}}$ is open and dense in the closed set $\cap_{1\leq i\leq k} S_{e_i}$. Then $\overline{S_{\mathcal{C}}} = \cap_{1\leq i\leq k} S_{e_i}$. Once this is achieved, the local closeness of $S_{\mathcal{C}}$ follows.

Firstly, by deformation theory of smooth curves, the set $S^{sm}_{e_i}$ is open in S_{e_i}. Thus $\cap_{1\leq i\leq k} S^{sm}_{e_i}$ is open in $\cap_{1\leq i\leq k} S_{e_i}$.

Denote the difference $\cap_{1\leq i\leq k} S^{sm}_{e_i} - S_{\mathcal{C}}$ as A. We argue that its closure $\overline{A}$ is a higher codimension subset in $\cap_{1\leq i\leq k} S_{e_i}$.

At all $b \in A$, the class e_i, $1 \leq i \leq k$ are represented by irreducible exceptional curves. On the other hand, $A \cap S_{\mathcal{C}} = \emptyset$. Thus, the cones $\mathcal{EC}_b(C; Q)$, $b \in A$ have to be strictly larger than $\mathcal{C}$. In other words, for every $b \in A$, there must be some additional exceptional classes $\tilde{e} \in Q$, effective and irreducible over b. One may collect all such $\tilde{e} \in Q \in \mathcal{EC}_b(C; Q) - \mathcal{C}$ into a set E when b runs through all the points in A.

It is apparent that $A = \cap_{1\leq i\leq k} S^{sm}_{e_i} - S_{\mathcal{C}} \subset \cup_{\tilde{e}\in E}(\cap_{1\leq i\leq k} S_{e_i} \cap S_{\tilde{e}})$.

By condition (vii) of assumption 1, $(\cap_{1\leq i\leq k} S_{e_i} \cap S_{\tilde{e}})$ is of dimension $dim_{\mathbf{C}} B + \frac{1}{2}(\sum_{1\leq i\leq k} d_{GT}(e_i) + d_{GT}(\tilde{e}))$ and is of lower dimension than $\cap_{1\leq i\leq k} S_{e_i}$ because $d_{GT}(\tilde{e}) < 0$.

Because the set E is finite, $\cup_{\tilde{e}\in E}(\cap_{1\leq i\leq k} S_{e_i} \cap S_{\tilde{e}})$, a finite union of closed subsets of $\cap_{1\leq i\leq k} S_{e_i}$ is closed.

Thus, one may write $S_{\mathcal{C}}$ as

$$\cap_{1\leq i\leq k} S^{sm}_{e_i} - \cup_{\tilde{e}\in E}(\cap_{1\leq i\leq k} S_{e_i} \cap S_{\tilde{e}})$$

$$= \cap_{1\leq i\leq k} S_{e_i} - (\cap_{1\leq i\leq k} S_{e_i} - \cap_{1\leq i\leq k} S^{sm}_{e_i}) - \cup_{\tilde{e}\in E}(\cap_{1\leq i\leq k} S_{e_i} \cap S_{\tilde{e}}),$$

and apparently is open in $\cap_{1\leq i\leq k} S_{e_i}$. $\Box$

Because Q is a finite set, there are finitely many possible $\mathcal{C}$ with non-empty $S_{\mathcal{C}}$. By its definition, $S_{\mathcal{C}_1} \cap S_{\mathcal{C}_2} = \emptyset$ if $\mathcal{C}_1 \neq \mathcal{C}_2$.

Proposition 9 *Let $(S_{\mathcal{C}}, \mathcal{C})$ be a pair with $S_{\mathcal{C}} \neq \emptyset$. The boundary set of $S_{\mathcal{C}}$, $\overline{S_{\mathcal{C}}} - S_{\mathcal{C}}$, is contained inside a disjoint union of different $S_{\mathcal{C}'}$ with $\mathcal{C} \subset \mathcal{C}'$.*

Proof of proposition 9: Take an arbitrary $b \in \overline{S_{\mathcal{C}}} - S_{\mathcal{C}}$, there are three different possibilities.

(A). All the e_i, $1 \leq i \leq k$ which generates $\mathcal{C}$ still represent irreducible curves in $\mathcal{X}_b$. But some other new exceptional class in Q becomes effective over b and becomes a generator of $\mathcal{EC}_b(C;Q)$.

(B). The curves representing some or all of the e_i, $1 \leq i \leq k$ break into more than one irreducible components.

(C). Some new exceptional class $\tilde{e}$, $\tilde{e} \cdot e_i \geq 0$ becomes effective over b, while the curves representing some of the e_i break into more than one component. (The mixture of (A). and (B).)

Suppose that $\tilde{e}_j \in Q, 1 \leq j \leq l$ are the new exceptional class(es) in case (A)., consider the cone $\mathcal{C}'$ generated by $e_i, 1 \leq i \leq k$ and $\tilde{e}_j, 1 \leq j \leq \tilde{k}$. Such points b are contained in the set $S_{\mathcal{C}'}$ and apparently $\mathcal{C} \subset \mathcal{C}'$.

In case (B)., suppose that, say e_1, has broken into components, $e_1 = \sum_{1 \leq r \leq r_1} e_{1;r}$. Then for at least one $e_{1;r}$, $e_{1;r}$ pairs negatively with C, i.e. $e_{1;r} \cdot C < 0$. If not,

$$0 > e_1 \cdot C = (\sum_{1 \leq r \leq r_1} e_{1;r}) \cdot C = \sum_{1 \leq r \leq r_1} e_{1;r} \cdot C \geq 0,$$

contradicting to the condition (i). of the perfectness assumption. On the other hand, the condition (iv) of perfectness assumption 1 implies that $e_{1;r} \in Q$ if $e_{1;r} \cdot C < 0$. So $\mathcal{EC}_b(C;Q) \neq \emptyset$.

Such a b is in $S_{\mathcal{C}'}$ with $\mathcal{C}' = \mathcal{EC}_b(C;Q)$. Because b is in the closure of $S_{\mathcal{C}}$ over which the classes in $\mathcal{C}$ are effective, by the degeneration argument all classes in $\mathcal{C}$ remain effective over b as well. This implies that $\mathcal{C} \subset \mathcal{EC}_b(C;Q) = \mathcal{C}'$.

The discussion for the case (C). is similar and we leave it to the reader. $\Box$

The proposition motivates us to define a partial ordering among different $(S_{\mathcal{C}}, \mathcal{C})$.

Definition 13 *The pair $(S_{\mathcal{C}}, \mathcal{C})$ is said to be greater than $(S'_{\mathcal{C}}, \mathcal{C}')$ under $\succ$, denoted as $(S_{\mathcal{C}}, \mathcal{C}) \succ (S'_{\mathcal{C}}, \mathcal{C}')$, if $\mathcal{C} \subset \mathcal{C}'$.*

Notice that $\succ$ is a necessary condition for $\overline{S_{\mathcal{C}}} \cap S_{\mathcal{C}'}$ to be non-empty.

3.1 The Admissible Decomposition Classes over $S_{\mathcal{C}}$

Having addressed the structure of $S_{\mathcal{C}}$, we move ahead to address the decomposition classes.

Consider the disjoint union $\coprod_{\mathcal{C}, S_{\mathcal{C}} \neq \emptyset} S_{\mathcal{C}}$. Either it is equal to the whole B, or it is a closed subset (by proposition 9) of B. In the first case, some $S_{\mathcal{C}}$ is of top dimension in B and $\mathcal{C}$ is generated by a finite number of -1 classes in $\mathcal{C}$.

If it is the case, any effective curve dual to C in the fibers of the family $\mathcal{X} \mapsto B$ over $S_{\mathcal{C}}$ must break off a certain multiples of -1 curves and the family theory suffers the same symptom as in Mcduff's proposal [Mc] ($B = pt$).

The more interesting situation is when $\coprod_{\mathcal{C},S_{\mathcal{C}}\neq\emptyset} S_{\mathcal{C}}\subset B$ is a proper closed subset.

In such a situation, any effective curve dual to C over $b\in\coprod_{\mathcal{C},S_{\mathcal{C}}\neq\emptyset} S_{\mathcal{C}}$ has to break off certain multiples of exceptional curves in $\mathcal{EC}_b(C;Q)$. What types of exceptional curves it has to break off depends on which $S_{\mathcal{C}}$ does b lie in.

A sketch of the general enumerative application of our curve counting scheme to the family algebraic Seiberg-Witten invariants $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(1,C)$ can be outlined as the following.

(1). Break the whole base space B into different $S_{\mathcal{C}}$ and $B-\coprod_{\mathcal{C};S_{\mathcal{C}}\neq\emptyset} S_{\mathcal{C}}$.

This set level decomposition gives a stratification of B and $B-\coprod_{\mathcal{C},S_{\mathcal{C}}\neq\emptyset} S_{\mathcal{C}}$ is the only top dimensional stratum. The dimension of each $S_{\mathcal{C}}$ can be calculated through the argument of proposition 8.

(2). Attach a local family invariant contribution to each $\overline{S_{\mathcal{C}}}\neq\emptyset$ depending on the generators $e_i, 1\leq i\leq k$ of $\mathcal{C}$ and the breaking of C into $C-\sum_{1\leq i\leq k} e_i$ and $\sum_{1\leq i\leq k} e_i$. (see [Liu1] and [Liu3] for some examples) The invariant contribution is some mixed family invariant of the form $\mathcal{AFSW}_{\mathcal{X}\times_B\overline{S_{\mathcal{C}}}\mapsto\overline{S_{\mathcal{C}}}}(\cdot, C-\sum_{1\leq i\leq k} e_i)$.

We expect the local family invariant contribution to be nonzero only if $dim_{\mathbf{C}}B+\frac{1}{2}d_{GT}(C-\sum_{1\leq i\leq k} e_i)+\frac{1}{2}\sum_{1\leq i\leq k} d_{GT}(e_i)\geq dim_{\mathbf{C}}B+d_{GT}(C)$.

(3). Apply the family switching formula [Liu3] to the local family invariant contribution and identify it with certain mixed family invariant of $C-\sum_{1\leq i\leq k} n_ie_i$ over $\overline{S_{\mathcal{C}}}$. The changing of the multiplicities of e_i, $(1,1,1,\cdots,1)$ to $(n_1,n_2,n_3,\cdots,n_k)$ is called the switching process which has been discussed in section 2.

(4). As different $\overline{S_{\mathcal{C}}}$ may intersect, the naive subtraction of all the local family invariant contributions from the original $\mathcal{AFSW}_{\mathcal{X}\mapsto B}(1,C)$ leads to over-subtraction.

Inductively, one has to define a version of modified family invariant (see section 5.3. of [Liu1] for some explicit examples in the differentiable category) for each $S_{\mathcal{C}}$. Schematically it involves subtracting the modified family invariants (which have already been defined by the induction hypothesis) of $S_{\mathcal{C}'}$, $\overline{S_{\mathcal{C}}}\supset S_{\mathcal{C}'}$ to define the modified family invariant of $S_{\mathcal{C}}$.

(5). The inductive scheme passes through $B-\coprod_{\mathcal{C}} S_{\mathcal{C}}$ and finally one may define a modified family invariant of $B-\coprod_{\mathcal{C}} S_{\mathcal{C}}$ by subtracting all the modified family invariants of $S_{\mathcal{C}}$ defined. In the algebraic category, it involves the usage of residual intersection theory in [F] in arguing that the whole family moduli space of C over B, $\mathcal{M}_C\mapsto B$, can be separated into the subscheme over $\coprod_{\mathcal{C}} S_{\mathcal{C}}$ and the residual portion (which definition involves blowing ups inductively). The modified invariant of C is actually equal to the intersection number defined by the residual intersection theory. (See [Liu1] for a discussion in the $\mathcal{C}^\infty$ category.)

Definition 14 *Let $(\mathcal{C},S_{\mathcal{C}})$ be a pair and let e_i be the generators of $\mathcal{C}$. Suppose that the decomposition $(C-\sum_{i\leq k} e_i, \sum_{i\leq k} e_i)$ of C satisfies the dimension inequality*

$$d_{GT}(C-\sum_{i\leq k}e_i)+\sum_{i\leq k}d_{GT}(e_i)\geq d_{GT}(C),$$

then we consider all the $k-$*tuples* $(n_1, n_2, \cdots, n_k)$, $n_i \in \mathbf{N}$ *such that*

$$d_{GT}(C-\sum_{i\leq k}n_ie_i)+\sum_{i\leq k}d_{GT}(e_i)\geq d_{GT}(C).$$

Moreover, there exists at least one i_0 *with* $1\leq i_0\leq k$, $n_{i_0}\geq 2$ *such that*

$$d_{GT}(C-\sum_{i\leq k; i\neq i_0}n_ie_i-(n_{i_0}-1)e_{i_0})+\sum_{i\leq k}d_{GT}(e_i)\geq d_{GT}(C).$$

All such $k-$*tuples define decompositions* $(C-\sum_{i\leq k}n_ie_i, \sum_{i\leq k}n_ie_i)$ *which can be derived from* $(C-\sum_{i\leq k}e_i, \sum_{i\leq k}e_i)$ *through elementary effective moves maintaining their family dimensions above the lower bound* $\frac{1}{2}d_{GT}(C)+p_g+dim_{\mathbf{C}}B$*. The set of all such decompositions forms an equivalence class called the admissible decomposition class associated to* $(\mathcal{C}, S_{\mathcal{C}})$*.*

The family switching formula [Liu3] allows us to relate the family invariants of different $(C-\sum_{i\leq k}n_ie_i, \sum_{i\leq k}n_ie_i)$ over $\overline{S_{\mathcal{C}}}$. That is why we view these different decompositions as equivalent.

In the published long paper [Liu1], one takes the universal spaces projection $f_n : M_{n+1}\mapsto M_n$ of an algebraic surface M as the fiber bundle $\mathcal{X}\mapsto B$. Let C be a $(1,1)$ class on M, then the set Q has been implicitly chosen to be the set of all type I exceptional classes e with $e\cdot(C-\mathbf{M}(E)E)<0$. A discussion parallel to the five steps above has been developed in the $\mathcal{C}^{\infty}$ category. Please consult [Liu1] for the notations and the details.

Remark 5 *We have made the simplifying assumptions 1 to study the stratification of the base space* B*. If the conditions (ii)-(v) in assumption 1 are not satisfied, the existence loci of exceptional curves may not be of the right dimension and one has to work with the Kuranishi models of the exceptional class to construct the algebraic cycle class representing the moduli cycles.*

Moreover, the violation of the conditions (vi)-(vii) indicates that the moduli spaces of different exceptional classes may not intersect transversally in B*. In the algebraic category we may use the intersection cycle class of the moduli cycle classes to represent the co-existence of different exceptional classes. The theory will be developed else where.*

References

[GH] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*. New York: John Wiley and Sons, (1978).

[Liu1] A. K. Liu, *Family Blowup Formula, Admissible Graphs and the Enumeration of Singular Curves (I)*. Journal of Differential Geometry **56** pp381-579 (2001)

[Liu2] A. K. Liu, *The Family Blowup Formula of the Family Seiberg-Witten Invariants* preprint, DG0305294, (2003)

[Liu3] A. K. Liu, *The Family Switching Formula and the $-n$ Exceptional Rational Curves* preprint, (2003)

[Mc] D. Mcduff, *Lecture on Gromov Invariants for Symplectic Four Manifolds* Nato Advanced Science Institute Series C, Math. Phys Science, **488** pp175-210 (1997)

[Mc2] D. Mcduff, *The Local Behavior of Holomorphic Curves in Almost Complex* 4-*Manifolds* Journal of Differential Geometry **34** no.1. pp143-164 (1991)

[MS] D. Salamon and D. Mcduff, *J Holomorphic Curves and Quantum Cohomology*. University Lecture Series 6. A.M.S. Providence, RI., (1994).

[T1] C.H. Taubes. *$SW \to Gr$, from the Seiberg-Witten Equations to Pseudo-holomorphic Curves*. Journal of American Mathematical Society. **9** no. 3. (1996).

[T2] C.H. Taubes, *$Gr \to SW$, from Pseudo-holomorphic Curves to Seiberg-Witten Solutions*, Journal of Differential Geometry, **51**, pp203-334. (1999).

[T3] C.H. Taubes, *The Seiberg Witten Invariants and Gromov Invariant*, Mathematical Research Letters. **2** pp221-238, (1995).